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Habibulla Akhadkulov
Universiti Utara Malaysia

Waleed Abduljabbar
University of Fallujah

Abdu Atta
Universiti Utara Malaysia

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AN APPLICATION OF QUADRUPLE FIXED POINT THEOREMS TO A NONLINEAR SYSTEM OF MATRIX EQUATIONS

Habibulla Akhadkulov[1], **Waleed Khalid Abduljabbar**[1], **Abdu Mohammed Ali Atta**[1] **and Sokhobiddin Akhatkulov**[2]

[1]School of Quantitative Sciences
Universiti Utara Malaysia
CAS 06010, UUM
Sintok, Kedah Darul Aman
Malaysia

[2]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi, Selangor Darul Ehsan
Malaysia

Abstract

We provide a quadruple fixed point theorem and prove the existence and uniqueness of solution of a nonlinear system of matrix equations.

1. Introduction

Fixed point theory plays a crucial role in solving various types of problems in nonlinear analysis such as, differential equations, integral equations, matrix equation, etc. One of fundamental results of the fixed point

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theory is Banach contraction mapping principle [4]. Due to its importance in nonlinear analysis, this theorem has been generalized in many ways with regard to different abstract spaces. One of interesting ways on the generalization is to investigate multidimensional fixed points such as coupled, tripled, quadruple and ϒ-fixed points. It should be noted that many fixed point results are obtained concerning the notions of coupled, tripled, quadruple and ϒ-fixed points (see, for instance, [1, 3, 6, 8, 10]).

In this paper, we discuss on the applications of quadruple fixed point theorems. More precisely, using some quadruple fixed point theorems, we study the existence and uniqueness of a solution of the following nonlinear systems of matrix equations in Hermitian matrices space:

$$\begin{cases} X_1 = Q + A_1^*\overline{F}(X_1)A_1 - A_2^*\overline{F}(X_2)A_2 - A_3^*\overline{F}(X_3)A_3 + A_4^*\overline{F}(X_4)A_4, \\ X_2 = Q + A_1^*\overline{F}(X_1)A_1 - A_2^*\overline{F}(X_2)A_2 - A_3^*\overline{F}(X_3)A_3 + A_4^*\overline{F}(X_4)A_4, \\ X_3 = Q + A_1^*\overline{F}(X_1)A_1 - A_2^*\overline{F}(X_2)A_2 - A_3^*\overline{F}(X_3)A_3 + A_4^*\overline{F}(X_4)A_4, \\ X_4 = Q + A_1^*\overline{F}(X_1)A_1 - A_2^*\overline{F}(X_2)A_2 - A_3^*\overline{F}(X_3)A_3 + A_4^*\overline{F}(X_4)A_4, \end{cases} \tag{1}$$

where Q and A_i, $i = 1, 2, 3, 4$ are positive matrices and $\overline{F}$ is a matrix function defined on the set of Hermitian matrices. Next, we recall some necessary notions in order to formulate our main results. The notion of the quadruple fixed point was introduced by Karapinar and Luong in [7] as follows:

Definition 1.1. Let X be a nonempty set and $F : X^4 \to X$ be a given mapping. An element $(z, y, z, w) \in X^4$ is called a *quadruple fixed point* of F if $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$, $F(w, x, y, z) = w$.

However, after introducing the notion of ϒ-fixed point for the definition, see [8], the notion has been generalized. Further, we provide a generalized definition of quadruple fixed point using the notion of ϒ-fixed point. Let $\Lambda = \{1, 2, 3, 4\}$ and $\{A, B\}$ be a partition of Λ, i.e., $A \bigcup B = \Lambda$ and $A \bigcap B = \varnothing$.

Next, define two sets of mappings as follows:

$$\Omega_{A,B} = \{\sigma : \Lambda \to \Lambda : \sigma(A) \subseteq A,\ \sigma(B) \subseteq B\},$$

$$\Omega'_{A,B} = \{\sigma : \Lambda \to \Lambda : \sigma(A) \subseteq B,\ \sigma(B) \subseteq A\}.$$

Let $\Upsilon = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be a 4-tuple of mapping such that $\sigma_i : \Lambda \to \Lambda$ and $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$.

Definition 1.2. A point $x = (x_1, x_2, x_3, x_4) \in X^4$ is called a *quadruple fixed point* of a mapping $F : X^4 \to X$ if

$$F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, x_{\sigma_1(3)}, x_{\sigma_1(4)}) = x_1,\ F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, x_{\sigma_1(3)}, x_{\sigma_1(4)}) = x_2,$$

$$F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, x_{\sigma_1(3)}, x_{\sigma_1(4)}) = x_3,\ F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, x_{\sigma_1(3)}, x_{\sigma_1(4)}) = x_4.$$

One can easily see that in Definition 1.2, if we choose $\Upsilon = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ as

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) & \sigma_1(4) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) & \sigma_2(4) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) & \sigma_3(4) \\ \sigma_4(1) & \sigma_4(2) & \sigma_4(3) & \sigma_4(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \qquad (2)$$

then we get Definition 1.1. Here and further, we denote by $(X, d, \prec)$ a partially ordered metric space. Using the partially ordered metric space $(X, d, \prec)$ and partition $\{A, B\}$, we define a four dimensional partially ordered metric space $(X^4, d_4, \prec_4)$ as follows:

- The maximum metric $d_4 : X^4 \times X^4 \to [0, \infty)$ is given by

$$d_4(\bar{x}, \bar{y}) = \max_{1 \le i \le 4} d(x_i, y_i),$$

where $\bar{x} = (x_1, x_2, x_3, x_4)$, $\bar{y} = (y_1, y_2, y_3, y_4) \in X^4$.

- The partial order w.r.t $\{A, B\}$ that is, for any $\bar{x} = (x_1, x_2, x_3, x_4)$, $\bar{y} = (y_1, y_2, y_3, y_4)$, we have

$$\bar{x} \leq_4 \bar{y} \Leftrightarrow \begin{cases} x_i \leq y_i, \ i \in A, \\ x_i \geq y_i, \ i \in B. \end{cases}$$

It is easy to see that if (X, d) is a complete metric space, then (X^4, d_4) is also a complete metric space.

Definition 1.3. An ordered metric space $(X, d, \prec)$ is called *regular* if it satisfies the following:

- if $\{x_m\}$ is a nondecreasing sequence and $\{x_m\} \overset{d}{\to} x$, then $x_m \prec x$ for all m,

- if $\{y_m\}$ is a nonincreasing sequence and $\{y_m\} \overset{d}{\to} y$, then $y_m \succ y$ for all m.

Definition 1.4. We say that a mapping $F : X^4 \to X$ has the *mixed monotone property* w.r.t partition $\{A, B\}$ if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B.

2. Quadruple Fixed Point Theorems

In this section, we provide relations between one and quadruple fixed point theorems. Define $T_\Upsilon : X^4 \to X^4$ as follows:

$$T_\Upsilon(x_1, x_2, x_3, x_4) = (F(x_{\sigma_1(1)}, \ldots, x_{\sigma_1(4)}), \ldots, F(x_{\sigma_4(1)}, \ldots, x_{\sigma_4(4)}))$$

for all $\bar{x} = (x_1, \ldots, x_4) \in X^4$. The following theorem is a consequence of Theorem 3.2 in [9].

Theorem 2.1. *Let* $(X, d, \prec)$ *be a complete partially ordered metric space. Let* $\Upsilon : \Lambda \to \Lambda$ *be a four tuple of mapping* $\Upsilon = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ *which is verifying,* $\sigma_i \in \Omega_{A,B}$ *if* $i \in A$ *and* $\sigma_i \in \Omega'_{A,B}$ *if* $i \in B$.

- *If* F *has the mixed monotone property, then* T_Υ *is monotone nondecreasing w.r.t* $\prec_4$.

- *If F is continuous, then* T_{Υ} *is also continuous*.

- *A point* $\bar{x} = (x_1, x_2, x_3, x_4) \in X^4$ *is a* Υ*-fixed point of F if and only if* $\bar{x} = (x_1, x_2, x_3, x_4)$ *is a fixed point of* T_{Υ}.

Definition 2.2. A function $\psi : [0, \infty) \to [0, \infty)$ is called an *altering distance function*, if ψ is continuous, monotonically increasing and $\psi(0) = 0$.

The following theorem will be used in the proof of our main theorem:

Theorem 2.3. *Let* $(X, d, \prec)$ *be a complete partially ordered metric space*. *Let* $\Upsilon : \Lambda \to \Lambda$ *be a four tuple of mapping* $\Upsilon = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ *which is verifying*, $\sigma_i \in \Omega_{A, B}$ *if* $i \in A$ *and* $\sigma_i \in \Omega'_{A, B}$ *if* $i \in B$. *Suppose* $F : X^4 \to X$ *is a mapping which obeys the following conditions*:

(i) *there exist altering distance functions* ψ, θ *and a monotonically decreasing continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that for all* $\bar{x} = (x_1, x_2, x_3, x_4)$, $\bar{y} = (y_1, y_2, y_3, y_4) \in X^4$ *with* $\bar{x} \prec_4 \bar{y}$,

$$\psi(d(F(x_1, x_2, x_3, x_4), F(y_1, y_2, y_3, y_4))) \le \theta(d_4(\bar{x}, \bar{y})) - \varphi(d_4(\bar{x}, \bar{y})),$$

where $\theta(0) = \varphi(0) = 0$ *and* $\psi(x) - \theta(x) + \varphi(x) > 0$ *for all* $x > 0$,

(ii) *there exists* $\bar{x}^0 = (x_1^0, x_2^0, x_3^0, x_4^0) \in X^4$ *such that*

$$x_i^0 \prec F(x^0_{\sigma_1(1)}, x^0_{\sigma_1(2)}, x^0_{\sigma_1(3)}, x^0_{\sigma_1(4)}) \text{ if } i \in A$$

and

$$x_i^0 \succ F(x^0_{\sigma_1(1)}, x^0_{\sigma_1(2)}, x^0_{\sigma_1(3)}, x^0_{\sigma_1(4)}) \text{ if } i \in B,$$

(iii) *F has the mixed monotone property w.r.t* $\{A, B\}$,

(iv) *F is continuous or* $(X, d, \prec)$ *is regular*.

Then F has at least one quadruple fixed point. Moreover,

(v) *if for any* $\bar{x} = (x_1, x_2, x_3, x_4)$, $\bar{y} = (y_1, y_2, y_3, y_4) \in X^4$, *there exists* $\bar{z} = (z_1, z_2, z_3, z_4) \in X^4$ *such that* $\bar{x} \prec_4 \bar{z}$ *and* $\bar{y} \prec_4 \bar{z}$, *then F has a unique quadruple fixed point* $\bar{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*) \in X^4$.

Proof. Using condition (i), we get

$$\begin{aligned}
&\psi(d_4(T_\Upsilon(x), T_\Upsilon(y)))\\
&= \psi(\max_{i\in\Lambda} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, x_{\sigma_i(4)}),\\
&\quad F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, y_{\sigma_i(4)})))\\
&\le \max_{i\in\Lambda}(\theta(\max_{j\in\Lambda} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})) - \varphi(\max_{j\in\Lambda} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})))\\
&\le \theta(d_4(\bar{x}, \bar{y})) - \varphi(d_4(\bar{x}, \bar{y}))
\end{aligned}$$

for all $\bar{x} = (x_1, x_2, x_3, x_4)$, $\bar{y} = (y_1, y_2, y_3, y_4) \in X^4$ such that $\bar{x} \prec_4 \bar{z}$. Thus, we have shown that the mapping T_Υ satisfies the contractive condition of Theorem 2.5 in [10]. The rest of the proof follows exactly same way that of Theorem 2.5 in [10].

3. Main Theorem

In this section, we prove the existence and uniqueness of solutions of a nonlinear system of matrix equations. We deal on the set of $n \times n$ matrices and denote this set by $M(n)$. Let $H(n)$ be the set of all $n \times n$ Hermitian matrices, $P(n)$ be the set of all $n \times n$ positive definite matrices and $\widetilde{P}(n)$ be the set of all $n \times n$ positive semidefinite matrices. Our aim is to find solution $X_1, X_2, X_3, X_4 \in H(n)$ of the system (1). Let us first provide some necessary facts. A partial order on $H(n)$ is defined by

$$X, Y \in H(n), \quad X \prec Y \Leftrightarrow Y - X \in \widetilde{P}(n).$$

The set $H(n)$ is partially ordered and for every $X, Y \in H(n)$, there is a greatest lower bound and a least upper bound (see [2]). Next, we use the following two norms:

$$\| A \| = \sqrt{\lambda_{\max}(A^* A)} = \max_{1 \leq i \leq n} s_i(A) \text{ the spectral norm,}$$

$$\| A \|_1 = tr(\sqrt{A^* A}) = \sum_{i=1}^{n} s_i(A) \text{ the trace norm,}$$

where $s_i(A)$, $i = 1, 2, \ldots, n$ are the singular values of A and $tr(\cdot)$ is the trace of a matrix.

Further, it is convenient us to use metric induced by the trace norm. Since $H(n)$ is a finite dimensional linear metric space equipped with the metric indicated by $\| \cdot \|_1$, complete (see Theorem IX. 2.2 in [5]).

The following lemma plays a key role for our application:

Lemma 3.1 [2]**.** *Let* $A, B \in \widetilde{P}(n)$. *Then we have*

$$0 \leq tr(AB) \leq \| A \| tr(B),$$

where $\| \cdot \|$ *is the spectral norm*.

Now we provide some hypothesis for the system (1). For the system, we suppose:

(a) $A_2^* \overline{F}(Q) A_2 + A_3^* \overline{F}(Q) A_3 \prec Q$,

(b) $\overline{F}$ is continuous, $\overline{F}(0_n) = 0_n$ and preserves the order that is:

$$X \prec Y \Rightarrow \overline{F}(X) \prec \overline{F}(Y),$$

where 0_n is the $n \times n$ zero matrix,

(c) there exists a positive number M' such that

$$\| A_1 A_1^* \| + \| A_2 A_2^* \| + \| A_3 A_3^* \| + \| A_4 A_4^* \| \leq M',$$

(d) for any $X, Y \in H(n)$ such that $X \prec Y$, we have

$$| tr(\overline{F}(X) - \overline{F}(Y)) | \leq \frac{1}{M'}, \sqrt{\ln[(tr(X - Y))^2 + 1]}.$$

Our main theorem is the following:

Theorem 3.2. *Under assumptions* (*a*)-(*d*), *the system* (1) *has a unique solution in* $H(n)$.

Proof. Consider the partition $A = \{1, 4\}$ and $B = \{2, 3\}$ of $\Lambda = \{1, 2, 3, 4\}$.

We choose $\Upsilon = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ as follows:

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) & \sigma_1(4) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) & \sigma_2(4) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) & \sigma_3(4) \\ \sigma_4(1) & \sigma_4(2) & \sigma_4(3) & \sigma_4(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Now we consider an operator $\overline{B} : H^4(n) \to H(n)$ defined as

$$\overline{B}(X_1, X_2, X_3, X_4) = Q + A_1^*\overline{F}(X_1)A_1 + A_2^*\overline{F}(X_2)A_2 + A_3^*\overline{F}(X_3)A_3 + A_4^*\overline{F}(X_4)A_4.$$

It is clear that the system (1) has a solution if and only if $\overline{B}$ has a quadruple fixed point. Therefore, further we show that the operator $\overline{B}$ satisfies all conditions of Theorem 2.3. Since $\overline{F}$ is continuous, $\overline{B}$ is continuous.

Next, we show that $\overline{B}$ has the mixed monotone property w.r.t $\{A, B\}$. By assumption (b), the mapping $\overline{F}$ preserves order, therefore for any $(X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4) \in H^4(n)$ such that

$$(X_1, X_2, X_3, X_4) \prec_4 (Y_1, Y_2, Y_3, Y_4) \Leftrightarrow \begin{cases} X_1 \prec Y_1 \\ X_2 \succ Y_2 \\ X_3 \succ Y_3 \\ X_4 \prec Y_4, \end{cases}$$

we have

$$(\overline{F}(X_1),\ \overline{F}(X_2),\ \overline{F}(X_3),\ \overline{F}(X_4)) \prec_4 (\overline{F}(Y_1),\ \overline{F}(Y_2),\ \overline{F}(Y_3),\ \overline{F}(Y_4))$$

$$\Leftrightarrow \begin{cases} \overline{F}(X_1) \prec \overline{F}(Y_1) \\ \overline{F}(X_2) \succ \overline{F}(Y_2) \\ \overline{F}(X_3) \succ \overline{F}(Y_3) \\ \overline{F}(X_4) \prec \overline{F}(Y_4). \end{cases}$$

Thus,

$$\overline{B}(Y_1, Y_2, Y_3, Y_4) - \overline{B}(X_1, X_2, X_3, X_4)$$

$$= A_1^*(\overline{F}(Y_1) - F(X_1))A_1 + A_2^*(\overline{F}(Y_2) - F(X_2))A_2$$

$$+ A_3^*(\overline{F}(X_3) - \overline{F}(Y_3))A_3 + A_4^*(\overline{F}(X_4) - \overline{F}(X_4))A_4 \succ 0_n.$$

Let $(Z_1^0, Z_2^0, Z_3^0, Z_4^0) = (Q, 0_n, 0_n, Q)$. Next, we show that

$$Z_i^0 \prec \overline{B}(Z_{\sigma_i(1)}^0, Z_{\sigma_i(2)}^0, Z_{\sigma_i(3)}^0, Z_{\sigma_i(4)}^0) \text{ if } i \in A,$$

$$Z_i^0 \succ \overline{B}(Z_{\sigma_i(1)}^0, Z_{\sigma_i(2)}^0, Z_{\sigma_i(3)}^0, Z_{\sigma_i(4)}^0) \text{ if } i \in B.$$

Indeed,

$$Q \prec Q + A_1^*\overline{F}(Q)A_1 + A_4^*\overline{F}(Q)A_4 = \overline{B}(Q, 0_n, 0_n, Q)$$

and by assumption (a), we have

$$\overline{B}(0_n, Q, Q, 0_n) = Q - A_2^*\overline{F}(Q)A_2 - A_3^*\overline{F}(Q)A_3 \succ 0_n.$$

Further, we show that $\overline{B}$ satisfies the first condition of Theorem 2.3 with $\psi(x) = x^2$, $\theta(x) = \ln(x^2 + 1)$ and $\varphi(x) = 0$. Let

$$(X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4) \in H^4(n)$$

such that $(X_1, X_2, X_3, X_4) \prec (Y_1, Y_2, Y_3, Y_4)$.

Since $\overline{B}(X_1, X_2, X_3, X_4) \prec \overline{B}(Y_1, Y_2, Y_3, Y_4)$, we have

$$\| \overline{B}(X_1, X_2, X_3, X_4) - \overline{B}(Y_1, Y_2, Y_3, Y_4) \|_1$$

$$= tr(A_1^*(\overline{F}(Y_1) - \overline{F}(X_1))A_1)$$

$$+ tr(A_2^*(\overline{F}(X_2) - \overline{F}(Y_2))A_2) + tr(A_3^*(\overline{F}(X_3) - \overline{F}(Y_3))A_3)$$

$$+ tr(A_4^*(\overline{F}(Y_4) - \overline{F}(X_4))A_4) \leq \sum_{i=1}^{4} \| A_i A_i^* \| \| \overline{F}(X_i) - \overline{F}(Y_i) \|_1. \quad (3)$$

Applying assumptions (c) and (d), we get

$$\sum_{i=1}^{4} \| A_i A_i^* \| \| \overline{F}(X_i) - \overline{F}(Y_i) \|_1$$

$$\leq \frac{1}{M'} \sum_{i=1}^{4} \| A_i A_i^* \| \sqrt{\ln[(\max_{1 \leq i \leq 4} \| X_i - Y_i \|_1)^2 + 1]}$$

$$\leq \sqrt{\ln[(\max_{1 \leq i \leq 4} \| X_i - Y_i \|_1)^2 + 1]}.$$

This and inequality (3) imply

$$\| \overline{B}(X_1, X_2, X_3, X_4) - \overline{B}(Y_1, Y_2, Y_3, Y_4) \|_1^2 \leq \ln[(\max_{1 \leq i \leq 4} \| X_i - Y_i \|_1)^2 + 1].$$

Thus, we have shown that the operator $\overline{B}$ satisfies the contraction condition (i) of Theorem 2.3 with $\psi(x) = x^2$, $\theta(x) = \ln(x^2 + 1)$ and $\varphi(x) = 0$. Hence, the operator $\overline{B}$ satisfies conditions (i)-(iv) of Theorem 2.3 and therefore it has a Υ-fixed point $(\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) \in H^4(n)$.

On the other hand, for all $X, Y \in H(n)$, there are greatest lower bound and least upper bound, hence all conditions of Theorem 2.3 are fulfilled. Therefore, $\overline{B}$ has a unique Υ-fixed point $(\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) \in H^4(n)$ which is also the unique solution of the system (1). This proves Theorem 3.2.

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References

[1] M. Abbas, A. R. Khan and T. Nazir, Coupled common fixed point results in two generalized metric spaces, Appl. Math. Comput. 217(13) (2011), 6328-6336.

[2] Andre C. M. Ran and Martine C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132(5) (2003), 1435-1443.

[3] H. Aydi, E. Karapinar and M. Postolache, Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces, Fixed Point Theory Appl. 1 (2012), 1-12.

[4] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922), 133-181.

[5] I. Gohberg and S. Goldberg, Basic Operator Theory, Birkhauser, Boston, MA, 1981.

[6] E. Karapinar, Quadruple fixed point theorems for weak ψ-contractions, ISRN Mathematical Analysis 2011 (2011), Article ID 989423, 15 pp.

[7] E. Karapinar and N. V. Luong, Quadruple fixed point theorems for nonlinear contractions, Comput. Math. Appl. 64(6) (2012), 1839-1848.

[8] A. Roldan, J. Martinez-Moreno and C. Roldan, Multidimensional fixed point theorems in partially ordered complete metric spaces, J. Math. Anal. Appl. 396(2) (2012), 536-545.

[9] A. Roldan, J. Martinez-Moreno, C. Roldan and E. Karapinar, Some remarks on multidimensional fixed point theorem, Fixed Point Theory 15(2) (2014), 545-558.

[10] F. Shaddad, M. S. Noorani, S. M. Alsulami and H. Akhadkulov, Coupled point results in partially ordered metric spaces without compatibility, Fixed Point Theory Appl. 2014, 2014:204, 18 pp.